CLOSED PROJECTIONS AND PEAK INTERPOLATION FOR OPERATOR ALGEBRAS

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ABSTRACT. The closed one-sided ideals of a C^*-algebra are exactly the closed subspaces supported by the orthogonal complement of a closed projection. Let A be a (not necessarily selfadjoint) subalgebra of a unital C^*-algebra B which contains the unit of B. Here we characterize the right ideals of A with left contractive approximate identity as those subspaces of A supported by the orthogonal complement of a closed projection in B^{**} which also lies in $A^{\perp\perp}$. Although this seems quite natural, the proof requires a set of new techniques which may may be viewed as a noncommutative version of the subject of peak interpolation from the theory of function spaces. Thus, the right ideals with left approximate identity are closely related to a type of peaking phenomena in the algebra. In this direction, we introduce a class of closed projections which generalizes the notion of a peak set in the theory of uniform algebras to the world of operator algebras and operator spaces.

1. INTRODUCTION

Let K be a compact Hausdorff space and let $C(K)$ denote the C^*-algebra of all complex-valued continuous functions on K. It is well known that closed ideals in $C(K)$ consist of all functions which vanish on a fixed closed subset of K. If instead, A is a uniform algebra contained in $C(K)$, then by a theorem of Hirsberg [17], a closed subspace J of A is a closed ideal with contractive approximate identity if and only if it consists of all functions which vanish on a 'p-set' for A. Recall, a subset E of K is said to be a *peak set* for A if there exists a function f in A such that $f(x) = 1$ for all $x \in E$ and $|f(x)| < 1$ for all $x \in E^c$. A subset E of K is said to be a *p-set* for A if it is the intersection of a family of peak sets. The p-sets for a uniform algebra A were characterized by Glicksberg [14] as those closed subsets E such that $\mu \in A^{\perp}$ implies $\mu_E \in A^{\perp}$. See [13], [14], or [18] for more information on peak sets.

For general C^*-algebras, the closed right ideals of a C^*-algebra A consist of the elements a in A for which $qa = 0$ for a *closed* projection q in the second dual of A. In other words, a subspace J is a right ideal of A if and only if

$$J = (1 - q)A^{**} \cap A,$$

for a closed projection q in A^{**}. Of course we are viewing A as being canonically embedded in its second dual, which is a W^*-algebra. In fact, $1 - q$ will be a weak*-limit point for any left contractive approximate identity of J. Indeed all closed projections arise in this manner.

Turning to the nonselfadjoint case, let A be a subalgebra of a unital C^*-algebra B, such that A contains the identity of B. We characterize the right ideals of A with left contractive approximate identity as those subspaces J of the form $J = (1-q)A^{**} \cap A$, for a closed (with respect to B^{**}) projection q in A^{**}. However natural this may

appear, the tools available in the selfadjoint theory are not applicable here. Thus a portion of this paper develops some technical tools from which this characterization follows. Incidentally, these generalize some peak interpolation results in the theory of uniform algebras. The above mentioned characterization is a refinement of the characterization in [6], which is in terms of right M-ideals. In particular, it appears to open up a new area in the theory of nonselfadjoint operator algebras, allowing for the generalization of certain important parts of the theory of C^*-algebras. This will be explored more fully in the sequel [7] where, for example, we apply the main result of this paper to develop a theory of hereditary subalgebras of not necessarily selfadjoint operator algebras. As is the case in the selfajoint theory, we demonstrate that these hereditary subalgebras are connected to the facial structure of the state space. Additionally, we also give a solution there to a more than ten year old problem in the theory of operator modules.

In our noncommutative setting, the peak and p-sets described above are replaced with a certain class of projections in the second dual of B, called the *peak* or *p-projections for A*. In the commutative case this class of projections can be identified with the characteristic functions of peak or p-sets for A. When $A = B$, the p-projections are exactly the closed projections in A^{**}. The theory of these projections brings another tool from the classical theory to the world of operator spaces.

The paper is organized as follows. In Section 2 we introduce the notation and discuss some background and preliminary results. In particular, we discuss the noncommutative topology of open and closed projections. Section 3 generalizes some interpolation results from the theory of function spaces to operator spaces, which will be used in Sections 4 and 5. Section 4 contains the main theorem and its proof. Finally, in Section 5 we look at closed projections in the weak*-closure of an operator algebra from the perspective of 'peak phenomena.'

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2. Open and closed projections and preliminary results

The theory of operator spaces and completely bounded maps has long been recognized as the appropriate setting for studying many problems in operator algebras. Basics on operator spaces may be found in [8], [11], [20], and [22]. We will make use of the following lemma which gives a criteria for when a completely bounded map is a complete isometry.

Lemma 2.1. *Let X be an operator space and Y a (not necessarily closed) subspace of another operator space. Suppose $T : X \to Y$ is a one-to-one and surjective completely bounded map such that T^* is a complete isometry. Then T is a complete isometry.*

Proof. Let Z be the closure of Y and define R to be the same as T except with range Z. Since $Z^* = Y^*$, we have that $R^* : Z^* \to X^*$ is simply T^* which is one-to-one and has closed range. By VI.6.3 in [10] this implies that R is onto. Since R

is onto, $Y = Z$ and $R = T$, and by the open mapping theorem T is bicontinuous. Hence, given $\varphi \in X^*$, then $\psi \equiv \varphi \circ T^{-1} \in Y^*$ satisfies $T^*\psi = \varphi$, showing that T^* is surjective and thus, since it is also completely isometric, T^{**} is a complete isometry. Viewing X and Y as being canonically embedded in their second duals, T is just the restriction of T^{**} to X. Hence, T is also a complete isometry. $\square$

Throughout this paper, B will denote a unital C^*-algebra and A will denote either a unital subspace or a unital subalgebra of B, for which by *unital* we mean $1_B \in A$. We view A and B as being canonically embedded into the second dual B^{**} of B via the canonical isometry. The second dual of B, B^{**}, is a W^*-algebra. By the *state space of* B, which we denote $S(B)$, we mean the set of positive functionals on B which have norm one. Each functional φ of B extends uniquely to a weak*-continuous, or *normal*, functional on B^{**}, which we again denote by φ. Also, we denote the unit in B by 1_B, or more often, simply by 1.

By a *projection* in B or B^{**} we mean an orthogonal projection. The *meet* of any two projections p and q can be given abstractly as

$$p \wedge q = \lim_{n\to\infty} (pq)^n,$$

where this limit is taken in the weak* topology. Similarly, the join is given by

$$p \vee q = \lim_{n\to\infty} 1 - (1 - p - q + pq)^n.$$

Let M be a (not necessarily selfadjoint) weak*-closed unital subalgebra of a W^*-algebra and suppose that p and q are projections in M. Then by the formula for $p \wedge q$ above, $p \wedge q$ is also in M. By induction, this extends to the meet of any finite collection of projections M. More generally, if $\{p_\alpha\}$ is any collection of projections, then $\wedge p_\alpha$ is the weak*-limit of the net of meets of finite subcollections of $\{p_\alpha\}$, each of which is in M. Thus $\wedge p_\alpha$ is also in M. Similarly, a join of projections in M is also in M. Now let $\pi : M \to B(H)$ be a weak*-continuous homomorphism of M into the bounded linear operators on a Hilbert space H. If p and q are projections in M, then $\pi((pq)^n) = (\pi(p)\pi(q))^n$ for each n, and so we have $\pi(p \wedge q) = \pi(p) \wedge \pi(q)$. This clearly generalizes to meets of finitely many projections. By approximating by such finite meets, this in turn generalizes to arbitrary meets of projections. Similar statements apply to joins of projections.

A projection $p \in B^{**}$ is said to be *open* if it is the weak*-limit of an increasing net (b_t) of elements in B with $0 \leq b_t \leq 1$. A projection $q \in B^{**}$ is said to be *closed* if $1 - q$ is open. It is clear that a closed projection is the weak*-limit of a decreasing net of positive elements in B. It is well known that a projection p in B^{**} is open if and only if it is the support of a left (respectively, right) ideal in B. That is, there exists a left (respectively, right) ideal J in B such that $J = B^{**}p \cap B$ (respectively, $J = pB^{**} \cap B$). In this case, the weak* closure of J in B^{**} is $B^{**}p$ (respectively, pB^{**}). Moreover, p is a weak*-limit point of any increasing right contractive approximate identity for J. Actually, if $p \in B^{**}$ is a projection which is a weak* limit of a net (e_t) in B such that $e_t p = e_t$, then p is open. To see this, we let J be the set of all $b \in B$ such that $bp = b$. Then J contains (e_t) and so p is in $J^{\perp\perp}$, which is a weak*-closed left ideal of B^{**}. Thus $B^{**}p \subset J^{\perp\perp}$, but also $J^{\perp\perp} \subset B^{**}p$, so that $J^{\perp\perp} = B^{**}p$. However, $J = B^{**}p \cap B$, so that p is the support of a closed left ideal, making it an open projection. A similar argument using right ideals holds if $pe_t = e_t$ instead. In the case that B is commutative, open and closed projections correspond to characteristic functions of open and closed sets,

respectively. It is this collection of open and closed projections which will act as a kind of substitute for topological arguments in the noncommutative situation. We now list, most without proof, some basic facts regarding these open and closed projections. Many of these facts can be found in Akemann's papers [1] and [2], and some may also be found in [21] and [15].

The join of any collection of open projections is again an open projection. Hence, the meet of any collection of closed projections is again a closed projection. However, in contrast to the commutative situation, joins of closed projections are not necessarily closed (see [1]). For a general C^*-algebra B, the presence of a unit guarantees a kind of noncommutative compactness. That is, if q is a closed projection, given any collection of open projections $\{p_\alpha\}$ such that $q \leq \bigvee_\alpha p_\alpha$, then there exists a finite subcollection $\{p_{\alpha_1}, \ldots, p_{\alpha_k}\}$ such that $q \leq \bigvee_{i=1}^k p_{\alpha_i}$ (see Proposition II.10 in [1]). We will refer to this as the 'compactness property.' A type of regularity also holds with respect to open and closed projections. Namely, any closed projection is the meet of all open projections dominating it. The following was communicated to us by Akemann.

Proposition 2.2. *Let q be a closed projection in B^{**}. Then*

$$q = \bigwedge\{u \mid u \geq q \textit{ and open}\}.$$

Proof. Assume that B and B^{**} are represented in the universal representation of B. Since q is closed we may find an increasing net (a_t) in B_{sa} such that $1 - a_t \searrow q$ weak* and $(1 - a_t)q = q$. Using the Borel functional calculus, for each t let

$$r_t = \chi_{(\frac{1}{2},\infty)}(1 - a_t),$$

where $\chi_{(\frac{1}{2},\infty)}$ is the characteristic function of the open interval $(\frac{1}{2}, \infty)$. Then necessarily r_t is an open projection such that

$$2(1 - a_t) \geq r_t.$$

Furthermore, we claim that each r_t dominates q. To prove this claim, fix t and let $\{f_n\}$ be an increasing sequence of positive continuous functions on the spectrum of $1 - a_t$ which converges point-wise to $\chi_{(\frac{1}{2},\infty)}$ and is such that $f_n(1) = 1$ for each n. Now fix n and suppose $\xi \in \operatorname{Ran} q$ is of norm one. Then $(1-a_t)\xi = (1-a_t)q\xi = q\xi = \xi$. So for any polynomial R, we have $R(1 - a_t)\xi = R(1)\xi$. Let R_k be a sequence of polynomials converging uniformly to f_n. Then

$$\langle f_n(1 - a_t)\xi, \xi\rangle = \lim_k \langle R_k(1)\xi, \xi\rangle = \lim_k R_k(1) = f_n(1) = 1.$$

By the converse to the Cauchy-Schwarz inequality, we have $f_n(1 - a_t)\xi = \xi$ for all $\xi \in \operatorname{Ran} q$, which is to say that $f_n(1 - a_t)q = q$. Hence $\chi_{(\frac{1}{2},\infty)}(1 - a_t)q = q$, and so $r_t \geq q$. Let $r_0 = \bigwedge_t r_t$ and suppose that r_0 does not equal q. Then there exists a state $\varphi \in S(B)$ such that $\varphi(r_0 - q) = 1$. This forces $\varphi(r_0) = 1$ and $\varphi(q) = 0$ since $r_0 - q \geq 0$. Thus

$$\varphi\left(2(1 - a_t)\right) \to 0.$$

However, since $\varphi(r_0) = 1$ and $r_t \geq r_0$, it must be that $\varphi(r_t) = 1$. Applying φ to the inequality

$$r_t \leq 2(1 - a_t)$$

and taking the weak* limit, we get $1 \leq 0$. Hence, $r_0 = q$ which proves the result. $\square$

Finally, one of the most important results in basic topology is Urysohn's lemma. Akemann has extended this result to closed projections:

Theorem 2.3 ([2]). *Let p and q be closed projections in B^{**} for a C^*-algebra B such that $pq = 0$. Then there exists an element a in B, $0 \leq a \leq 1$, such that $ap = p$ and $aq = 0$.*

We will often be working with closed projections in B^{**} which lie in the weak* closure of A in B^{**}. The following gives some equivalent conditions for this.

Lemma 2.4. *Let A be closed subalgebra of a unital C^*-algebra $B \subset B^{**}$ such that A contains the unit of B. Let $q \in B^{**}$ be a projection. The following are equivalent:*

(1) $q \in \overline{A}^{w*}$,
(2) $q \in A^{\perp\perp}$,
(3) $A^{\perp} \subset (qA)_{\perp}$.

Proof. The equivalence of (1) and (2) is a standard result of functional analysis. Suppose (3) holds. Then $((qA)_{\perp})^{\perp} \subset A^{\perp\perp}$. However, $((qA)_{\perp})^{\perp} = \overline{qA}^{w^*} = q\overline{A}^{w^*}$ which must contain q since A is unital. Hence, (2) holds. Now assume (2). By hypothesis, $\psi(q) = 0$ for all $\psi \in A^{\perp}$. Let $\varphi \in A^{\perp}$. Then for each $a \in A$, $\varphi(\cdot a) \in A^{\perp}$. Thus $\varphi(qa) = 0$ for all $a \in A$. Hence $\varphi \in (qA)_{\perp}$. $\square$

A class of operators which will play an important role here are the *completely non-unitary*, or *c.n.u.*, operators on a Hilbert space H. A contraction T is said to be *completely non-unitary* if there exists no reducing subspace for T on which T acts unitarily. It is well known that if T is completely non-unitary, then $T^n \to 0$ in the weak operator topology on $B(H)$ as $n \to \infty$. See [12] and [19] for details.

If B is a unital C^*-algebra, we denote the self-adjoint part of B by B_{sa}. Kadison's 'function representation' says that B_{sa} may be represented as continuous affine functions on $S(B)$ via an order preserving linear isometry which extends weak*-continuously to B^{**}_{sa}, in such a way that B^{**}_{sa} is represented as bounded affine functions on $S(B)$. We say that an element b of B^{**}_{sa} is lower semi-continuous if its image under this representation is a lower semi-continuous function on $S(B)$ ([21]).

Lemma 2.5. *Let b be a positive, lower semi-continuous contraction in B^{**} for a C^*-algebra B and suppose $\varphi_0(b) = 0$ for some $\varphi_0 \in S(B)$. Then there exists a pure state of B which is zero at b.*

Proof. Let $K = \{\varphi \in S(B) \mid \varphi(b) = 0\}$. The set K is nonempty by hypothesis, and since ϕ and b are positive, we also have that $K = \{\varphi \in S(B) \mid \varphi(b) \leq 0\}$. Thus K is the complement of the set $\{\varphi \in S(B) \mid \varphi(b) > 0\}$ which is open in the weak* topology by the semi-continuity property of b. Thus K is weak*-closed in $S(B)$ and hence weak*-compact. It is also convex by a straight-forward calculation. Thus, K is well supplied with extreme points by the Krein-Milman theorem. Now suppose that $\varphi_1, \varphi_2 \in S(B)$, λ is a scalar in $(0,1)$ and that $\lambda\varphi_1 + (1-\lambda)\varphi_2 \in K$. Then $\lambda\varphi_1(b) + (1-\lambda)\varphi_2(b) = 0$. However, by positivity, this forces $\varphi_1(b) = \varphi_2(b) = 0$ and so φ_1 and φ_2 are in K. In other words, K is a face of $S(B)$ and hence must contain an extreme point of $S(B)$. However, the extreme points of $S(B)$ are the pure states of B. $\square$

3. Noncommutative peak interpolation

The following sequence of propositions and lemmas are the keys to the main result and generalize some classical results from the theory of function spaces (see Section II.12 of [13]).

Proposition 3.1. *Let X be a closed subspace of a C^*-algebra B. Let $q \in B^{**}$ be a projection such that $\varphi \in (qX)_\perp$ for all $\varphi \in X^\perp$. Let $I = \{x \in X : qx = 0\}$. Then qX is completely isometric to X/I via the map $x + I \mapsto qx$. Similarly, if I is defined to be $\{x \in X : xq = 0\}$, then Xq is completely isometric to X/I via the map $x + I \mapsto xq$.*

Proof. We will be using standard operator space duality theory, as may be found in [8], for example. First note that I is the kernel of the completely contractive map $x \mapsto qx$ on X, so that this map factors through the quotient X/I:

$$X \xrightarrow{S} X/I \xrightarrow{T} qX,$$

where S is the natural quotient map and T is the induced linear isomorphism. Taking adjoints, we have $(T \circ S)^* = S^* \circ T^*$ and if $\varphi \in (qX)^*$ and $x \in X$, then

$$(S^* \circ T^*)(\varphi)(x) = S^*(T^*\varphi)(x) = (T^*\varphi)(Sx) = \varphi(TSx) = \varphi(T(x + I)) = \varphi(qx),$$

so that $S^* \circ T^*$ is given by

$$\varphi \mapsto \varphi(q\cdot),$$

for each $\varphi \in (qX)^*$. Identifying $(qX)^*$ with $(qB)^*/(qX)^\perp$ and X^* with $B^*/X^\perp$, the map $S^* \circ T^*$ takes an element $\varphi + (qX)^\perp$ to the element $\varphi(q\cdot) + X^\perp$. To show that T is a complete isometry, by Lemma 2.1, it suffices to show that T^* is a complete isometry, since T is one-to-one, surjective, and completely bounded. Since S^* is completely contractive, if $S^* \circ T^*$ is completely isometric, then

$$\|\varphi\| = \|(S^* \circ T^*)(\varphi)\| \le \|T^*\varphi\| \le \|\varphi\|.$$

Similar statements also hold for each matrix level. Hence, if $S^* \circ T^*$ is completely isometric, then so is T^*. Thus, in order to show that T^* is completely isometric, it is sufficient to show that $S^* \circ T^*$ is a complete isometry. Note that since $T \circ S$ is completely contractive, so is $S^* \circ T^*$. We let (e_t) be a decreasing net in the unit ball of B, such that $e_t \to q$ weak* and $qe_t = q$ for all t.

Let $\varphi \in (qB)^*$ and $\psi \in X^\perp$. Then $\psi \in (qX)_\perp \subset (qX)^\perp$. If J is the right ideal in B supported by $1 - q$, then for $qb \in \mathrm{Ball}(qB)$, we have $\|qb\| = \|b + J\|$. Since right ideals are proximinal in a C^*-algebra, it follows that there exists $a \in J$ such that $\|qb\| = \|b + J\| = \|b + a\|$. Since $q(b + a) = qb$ and $\|b + a\| \le 1$, by replacing b with $b + a$ it follows that

$$\|\varphi + \psi\|_{(qB)^*} \quad = \quad \sup\{|\varphi(qb) + \psi(qb)| : b \in \mathrm{Ball}(B)\}.$$

However, for $b \in \mathrm{Ball}(B)$, we have

$$\begin{aligned} |\varphi(qb) + \psi(qb)| &= \lim_t |\varphi(qe_tb) + \psi(e_tb)| \\ &\le \|\varphi(q\cdot) + \psi\|_{B^*}\|e_tb\| \\ &\le \|\varphi(q\cdot) + \psi\|_{B^*}. \end{aligned}$$

Hence, $\|\varphi + \psi\|_{(qB)^*} \le \|\varphi(q\cdot) + \psi\|_{B^*}$, and, thus, $\|\varphi + (qX)^\perp\| \le \|\varphi(q\cdot) + \psi\|_{B^*}$. Now taking the infimum over all $\psi \in X^\perp$, we get $\|\varphi + (qX)^\perp\| \le \|\varphi(q\cdot) + X^\perp\|$.

The matricial case is almost identical, using operator space duality principles, and is left to the reader to fill in the details. The last statement of the proposition follows by a completely analogous proof. ☐

Since $qX \subset qB$ and qB can be identified with a quotient B/J, where J is the right ideal of B corresponding to q, then the result above shows that the set $\{x + J | x \in X\}$ is closed in B/J.

Proposition 3.2. *Let X be a closed subspace of a C^*-algebra B. Let $q \in B^{**}$ be a projection such that $\varphi \in (qX)_\perp$ whenever $\varphi \in X^\perp$. Let p be a strictly positive element in B and let $a \in X$ such that $a^*qa \leq p$. Given $\epsilon > 0$ there exists $b \in X$ such that $qb = qa$ and $b^*b \leq p + \epsilon 1_B$.*

Proof. First assume $p = 1$. Let $I = \{x \in X : qx = 0\}$ and let $\delta > 0$ such that $2\delta + \delta^2 < \epsilon$. Then by the previous lemma there exists an $h \in I$ such that $\|a + h\| \leq \|qa\| + \delta$. Let $b = a + h$ and note that $qb = qa$. Also, since $a^*qa \leq 1$, we have $\|qa\| \leq 1$. Then

$$\begin{aligned} b^*b &\leq \|b^*b\|1_B = \|b\|^2 1_B \\ &\leq (\|qa\| + \delta)^2 1_B \leq (1 + \delta)^2 1_B \\ &= (1 + 2\delta + \delta^2) 1_B \\ &\leq (1 + \epsilon) 1_B = p + \epsilon 1_B. \end{aligned}$$

In the case that p is not necessarily 1, note that $a^*qa \leq p$ is equivalent to

$$p^{-1/2} a^* q a p^{-1/2} \leq 1.$$

Furthermore, note that $p^{-1/2} a^* q a p^{-1/2} = (ap^{-1/2})^* q (ap^{-1/2})$. Now suppose that $\varphi \in (Xp^{-1/2})^\perp \subset B^*$. Then $\varphi(\cdot p^{-1/2}) \in X^\perp$. Hence, by hypothesis, $\varphi(\cdot p^{-1/2}) \in (qX)_\perp$, and thus $\varphi \in (qXp^{-1/2})^\perp$. So by the $p = 1$ case, there exists $bp^{-1/2} \in Xp^{-1/2}$ such that

$$qbp^{-1/2} = qap^{-1/2},$$

and

$$p^{-1/2} b^* b p^{-1/2} \leq 1 + \epsilon \|p\|^{-1}.$$

Pre- and post- multiplying by $p^{1/2}$ yields

$$b^*b \leq p + \epsilon \|p\|^{-1} p \leq p + \epsilon.$$

☐

Proposition 3.3. *Let X be a unital subspace of B and suppose q is a projection in B^{**} such that $\varphi \in (qX)_\perp$ for every $\varphi \in X^\perp$. Let p be a strictly positive contraction in B. If $a \in X$ with $a^*qa \leq p$, then there exists b in the unit ball of $\{x \in X : qx = qa\}^{\perp\perp}$ such that $b^*b \leq p$. Moreover, $qb = qa$.*

Proof. As in the previous proposition, we first show that the lemma holds in the case $p = 1$. Suppose $p = 1$. By the previous lemma, for each $n > 1$ there is a $b_n \in X$ such that $qb_n = qa$ and $b_n^* b_n \leq 1 + \frac{1}{n}$. By the weak*-compactness of $\mathrm{Ball}(\overline{X}^{w^*})$, (b_n) has a weak*-limit point b in $\mathrm{Ball}(\overline{X}^{w^*})$. Thus $b^*b \leq 1$. Let (b_{n_t}) be a subnet of (b_n) converging to b. Then by weak*-continuity we must also have that $qb = qa$. For general p, as before we note that $a^*qa \leq p$ is equivalent to

$$p^{-1/2} a^* q a p^{-1/2} \leq 1,$$

and that $p^{-1/2}a^*qap^{-1/2} = (ap^{-1/2})^*q(ap^{-1/2})$. Now let $\varphi \in (Xp^{-1/2})^\perp$. Then $\varphi(\cdot p^{-1/2}) \in X^\perp$. Thus, by hypothesis, $\varphi(\cdot p^{-1/2}) \in (qX)_\perp$, and thus $\varphi \in (qXp^{-1/2})^\perp$. So by the $p = 1$ case, there exists $bp^{-1/2} \in \overline{Xp^{-1/2}}^{w^*} = \overline{X}^{w^*} p^{-1/2}$ such that

$$qbp^{-1/2} = qap^{-1/2},$$

and

$$p^{-1/2}b^*bp^{-1/2} \leq 1.$$

We pre- and post- multiply by $p^{1/2}$ to get $b^*b \leq p$. $\square$

Remarks 1) The preceding two lemmas have matricial variants. For instance, the conclusion to Lemma 3.2 can be generalized to read 'for every strictly positive contraction $p \in M_n(B)$ and $a \in M_n(X)$ with $a^*(I_n \otimes q)a \leq p$, there exists $b \in M_n(X)$ such that $(I_n \otimes q)a = (I_n \otimes q)b$ and $b^*b \leq p + \epsilon I_n$.' Here I_n denotes the identity matrix in M_n.

2) If X is a reflexive unital subspace of B and q is such that $\varphi \in (qX)_\perp$ for every $\varphi \in X^\perp$, then for every strictly positive contraction $p \in B$ with $q \leq p$, there exists $a \in \text{Ball}(X)$ such that $qa = q$ and $a^*a \leq p$.

Variants of Propositions 3.2 and 3.3 in the commutative case are related to the subject of 'peak interpolation' from the theory of function algebras (see e.g. [13]). For example, suppose q above is such that $\varphi \in A^\perp$ implies that $\varphi(q\cdot) = 0$, where we view $\varphi(q\cdot)$ as an element of $(qB)^*$. Now suppose that $\psi_0 \in (qA)^\perp$, viewing $(qA)^\perp$ as a subspace of $(qB)^*$. Now define $\psi \in B^*$ by $\psi(b) = \psi_0(qb)$ for all $b \in B$. Then $\psi \in A^\perp$ and, so $\psi(q\cdot) = 0$ as a functional on qB, by hypothesis. Hence, we also have $\psi_0(qb) = 0$ for all $b \in A$. Thus $(qA)^\perp = \{0\}$ and so qA is norm dense in qB. However, by Lemma 3.1, qA is norm closed, and so $qA = qB$. Now let $\epsilon > 0$ and let p be a strictly positive element of B. Given $a \in B$ with $a^*qa \leq p$, by Lemma 3.2, there exists $b \in A$ such that $qb = qa$ and $b^*b \leq p + \epsilon$.

We close this section with several lemmas which are required in the remainder of the paper. The first one describes the weak*-limits of powers of certain types of contractions. The last two are useful tools for generating certain closed projections associated with a contraction.

Lemma 3.4. *Let B be a C^*-algebra and let a be a contraction in B^{**}. Let q be a projection in B^{**} such that*

(1) *$aq = q$, and*
(2) *$\varphi(a^*a) < 1$ for all $\varphi \in S(B)$ such that $\varphi(q) = 0$.*

Then (a^n) and $((a^*a)^n)$ converge weak to q as $n \to \infty$.*

Proof. We have that B is contained non-degenerately in $B(H)$, where H is the Hilbert space associated with the universal representation of B. We may also view B^{**} as a von Neumann algebra in $B(H)$. Let K be the range of q so that $B(H) = B(K \oplus K^\perp)$. With respect to this decomposition we may write

$$a = \begin{bmatrix} I_K & 0 \\ 0 & x \end{bmatrix},$$

where I_K is the identity operator on K and $x \in B(K^\perp)$. Let $\xi \in K^\perp$ be a unit vector. Let φ be the vector state corresponding to $0 \oplus \xi$. Then $\varphi(q) = 0$, so that $\varphi(a^*a) < 1$. Thus $\langle x\xi, x\xi \rangle < 1$ for any unit vector ξ in $K^\perp$. If x had a reducing

subspace on which x acted unitarily, then there would be a unit vector $\eta \in K^\perp$ such that $\langle x\eta, x\eta\rangle = 1$, which is a contradiction. Thus x must be completely non-unitary. This implies that $x^n \to 0$ in the weak operator topology as $n \to \infty$. Now let η_1 and η_2 be vectors in K and let ξ_1 and ξ_2 be vectors in $K^\perp$. Then,

$$\begin{aligned}\langle a^n(\eta_1 \oplus \xi_1), (\eta_2 \oplus \xi_2)\rangle &= \langle(\eta_1 \oplus x^n\xi_1), (\eta_2 \oplus \xi_1)\rangle \\ &= \langle \eta_1, \eta_2\rangle + \langle x^n\xi_1, \xi_2\rangle \to \langle \eta_1, \eta_2\rangle.\end{aligned}$$

Thus (a^n) converges to q. In order to show that $((a^*a)^n)$ also converges to q, it will suffice to show that x^*x is also c.n.u. For in this case, the same argument as above will also work. Suppose x^*x has a reducing subspace V on which it acts unitarily and let $\xi \in V$ be a unit vector. Then $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle < 1$, which contradicts x^*x acting unitarily on V. □

Lemma 3.5. *Let X be a unital subspace of B. Let $a \in Ball(X)$ and let q be a closed projection in B^{**} with $aq = q$. Define $b = \frac{1}{2}(a + 1)$. Then there exists a closed projection $r \in B^{**}$ such that $q \le r$ and satisfying*

1. *$br = r$, and*
2. *$\varphi(b^*b) < 1$ for all $\varphi \in S(B)$ such that $\varphi(r) = 0$.*

Proof. With $b = \frac{1}{2}(a + 1)$ and $aq = a$, it is clear that $bq = q$. This implies that $(1-b)(1-q) = 1-b$. Hence, $1-b \in B^{**}(1-q)\cap B$ and this contains the intersection, J, of all left ideals in B containing $1-b$. Let $p \in B^{**}$ be the support projection for the left ideal J. Then $(1-b)(1-p) = 0$, so that $b(1-p) = 1-p$. Hence $1-p$ satisfies condition (1). Now suppose that φ is a state of B such that $\varphi(1-p) = 0$. Then surely $\varphi(b^*b) \le 1$, but suppose that $\varphi(b^*b) = 1$. Then $\varphi(a^*a) + 2\mathrm{Re}\ \varphi(a) + 1 = 4$, which forces $\varphi(a^*a) = \varphi(a) = 1$, and hence, $\varphi(b) = 1$. Now let L denote the left kernel associated with φ. Then L is a left ideal and we claim that $1 - b \in L$. To see this, note that

$$\varphi((1-b)^*(1-b)) = \varphi(b^*b) - 2\mathrm{Re}\ \varphi(b) + 1 = 0.$$

Hence, $1 - b \in L$ and consequently, $J \subset L$. If p_L denotes the support projection of L, then by the definition of J, we must have $p \le p_L$ and so $1 - p_L \le 1 - p$. Applying φ to this last inequality yields $1 \le 0$, an obvious contradiction. Thus we conclude that $\varphi(b^*b) < 1$ and so $1 - p$ satisfies condition (2). Furthermore, since $J \subset B^{**}(1-q) \cap B$, it follows that $q \le 1 - p$. Now let $r = 1 - p$. □

We also need the following variant of Lemma 3.5:

Lemma 3.6. *Let a be a contraction in B^{**} and q a closed projection in B^{**} such that $aq = q$. Let $b = \frac{1}{2}(a + 1)$. Then there exists a projection $r \in B^{**}$ such that $r \ge q$ and satisfying*

1. *$br = r$, and*
2. *$\varphi(b^*b) < 1$ for all $\varphi \in S(B)$ such that $\varphi(r) = 0$.*

Moreover, $b^k \to r$ weak as $k \to \infty$.*

Proof. This is essentially the same proof as above. Let $b = \frac{1}{2}(a + 1)$ and let J be the intersection of all weak*-closed left ideals in B^{**} containing $1 - b$. This will be a weak*-closed left ideal. Let p be the support projection of J. Then, as above, $q \le 1-p$ and $b(1-p) = 1-p$. Now let $\varphi \in S(B)$ be such that $\varphi(1-p) = 0$. Letting L be the left-kernel of φ in B^{**}, we get a weak*-closed left ideal containing $1 - b$. As in the proof of Lemma 3.5, we let p_L be the support projection of L, so that

$1 - p_L \leq 1 - p$. Applying φ to this inequality yields the necessary contradiction. Taking $r = 1-p$ proves the first part. The second part follows from Lemma 3.4. $\square$

4. Right ideals with left contractive approximate identity

We now have everything needed to prove the main theorem. The following theorem gives the difficult direction.

Theorem 4.1. *Let A be a unital subalgebra of a unital C^*-algebra B. Let $q \in B^{**}$ be a closed projection such that $q \in A^{\perp\perp}$. Then $1 - q$ is in the weak*-closure of the right ideal $J = \{a \in A : (1-q)a = a\}$.*

Proof. First recall that $A^\perp \subset (qA)_\perp$ is equivalent to $q \in A^{\perp\perp}$ by Lemma 2.4. Thus q satisfies the hypotheses of Proposition 3.2 and 3.3. Let u be an open projection dominating q. Then, by the noncommutative Urysohn's lemma there exists $p \in B$, $0 \leq p \leq 1$, such that $pq = q$ and $p(1-u) = 0$. For each integer $n \geq 0$, let

$$p_n = \frac{n}{n+1}p + \frac{1}{n+1}.$$

Then p_n is strictly positive and $p_n q = q$, so that $q \leq p_n$. Each p_n also has the property that $p_n(1-u) = \frac{1}{n+1}(1-u)$. By Proposition 3.3, for each n there exists $a_n \in \mathrm{Ball}(A^{**})$ such that $qa_n = q$ and $a_n^* a_n \leq p_n$. The net (a_n) is contained in the unit ball of A^{**}, which is weak*-compact. Let (a_{n_t}) be a subnet converging to an element a in the unit ball of A^{**}. Since $(1-u)a_{n_t}^* a_{n_t}(1-u) \leq \frac{1}{n_t+1}$, the net $(1-u)a_{n_t}^* a_{n_t}(1-u)$ converges to zero in norm, and hence, by the C^*-identity, $a_{n_t}(1-u)$ also converges to zero in norm. It follows that $a(1-u) = 0$ and $au = a$. Similarly, $qa = q$. Let $b = \frac{1}{2}(a+1)$. From Lemma 3.6 we know that there is a projection $r \in B^{**}$ with $r \geq q$ such that $b^k \to r$ weak*. We now show that $r \leq u$. To do this, we first observe that

$$\begin{aligned} bu &= \left(\frac{1}{2}a + \frac{1}{2}\right)u \\ &= \frac{1}{2}a + \frac{1}{2}u \\ &= \frac{1}{2}a + \frac{1}{2}u + \frac{1}{2}(1-u) - \frac{1}{2}(1-u) \\ &= b - \frac{1}{2}(1-u), \end{aligned}$$

and so

$$b^k u = b^k - \frac{1}{2}b^{k-1}(1-u).$$

Thus in the weak*-limit, as $k \to \infty$, we get

$$ru = r - \frac{1}{2}r(1-u).$$

Hence,

$$2ru = 2r - r(1-u) = r + ru,$$

and therefore $ru = r$.

For each a_n, Lemma 3.6 gives rise to projections q_n in B^{**} with $q_n \geq q$ such that $b_n \equiv \frac{1}{2}(a_n + 1)$ has the following properties:

(1) $q_n b_n = q_n$,
(2) $\varphi(b_n^* b_n) < 1$ for all $\varphi \in S(B)$ such that $\varphi(q_n) = 0$, and

(3) $b_n^k \to q_n$ weak* as $k \to \infty$.

Item (1) implies that $1 - b_n$ is in J and item (3) implies that each q_n lies in A^{**}.

Now let $Q = \bigwedge q_n$, so that $Q \leq q_n$ for all n. Since $q_n \geq q$ we also have $Q \geq q$. The containment $B^{**}(1-q_n) \subset B^{**}(1-Q)$ follows from $Q \leq q_n$. However, $1 - b_n \in B^{**}(1-q_n)$ for each n. Hence $1 - b_n \in B^{**}(1-Q)$ for all n. However, $1 - b_{n_t} \to 1 - b$, and so $1 - b$ is in $B^{**}(1-Q)$. By the construction of r (recall $1 - r$ is the support projection for the weak*-closed left ideal generated by $1 - b$) this implies that $B^{**}(1-r) \subset B^{**}(1-Q)$. Therefore $q \leq Q \leq r \leq u$ and hence,

$$q \leq \bigwedge_n q_n \leq u.$$

Let $Q_u = \bigwedge_n q_n$, which is in A^{**}. As u varies over all open projections dominating q, we get

$$q \leq \bigwedge_{u \geq q} Q_u \leq \bigwedge_{u \geq q} u = q,$$

So,

$$q = \bigwedge_{u \geq q} Q_u = \bigwedge_{u \geq q} \bigwedge_n q_n.$$

Thus,

$$\begin{aligned} 1 - q &= 1 - \bigwedge_{u \geq q} \bigwedge_n q_n \\ &= \bigvee_{u \geq q} \left(1 - \bigwedge_n q_n \right) \\ &= \bigvee_{u \geq q} \bigvee_n (1 - q_n). \end{aligned}$$

If u is fixed, then for each q_n associated with u, $b_n q = q$, where $b_n \in A^{**}$, as above. From Proposition 3.3, each a_n is a weak*-limit of elements y in A satisfying $qy = q$. So each b_n is the weak*-limit of a net, (c_t) say, in A such that $qc_t = q$. Thus $(1-q)(1-c_t) = 1 - c_t$ and therefore the net $(1 - c_t)$ is contained in J. Hence its weak*-limit, $1 - b_n$, is in $J^{\perp\perp}$. Also, for any integer $k > 0$, $1 - b_n^k$ is in $J^{\perp\perp}$. Hence $1 - q_n = w^*\text{-}\lim_k 1 - b_n^k$ is in $J^{\perp\perp}$. Combining this last fact with the last displayed equation we see that $1 - q$ is in $J^{\perp\perp}$. $\square$

As a consequence, we have our main theorem characterizing right ideals with left contractive approximate identity.

Theorem 4.2. *Let A be a unital subalgebra of a unital C^*-algebra B. A subspace J of A is a right ideal with left contractive approximate identity if and only if $J = (1-q)A^{**} \cap A$ for a closed projection $q \in A^{\perp\perp}$.*

Proof. The forward implication is the easy direction and is essentially in [6]. Suppose J is a right ideal with left contractive approximate identity (e_t). Then $J^{\perp\perp}$ has a left identity p such that $e_t \to p$ weak* (see e.g. 2.5.8 in [8]). Since p is a contractive idempotent, it is an orthogonal projection. Since $pe_t = e_t$, p is an open projection by the discussion on open projections in Section 2. So $q = 1 - p$ is closed. Also, $J \subset (1-q)A^{**} \cap A$. However, if $a \in A$ such that $(1-q)a = a$, then $e_t a \in J$ and so $a = (1-q)a \in J^{\perp\perp} \cap A = J$. Thus $(1-q)A^{**} \cap A \subset J$. Now let $\varphi \in A^{\perp}$. Then $0 = \varphi(1 - e_t) \to \varphi(q)$, and so $\varphi(q) = 0$. Hence $q \in A^{\perp\perp}$.

On the other hand, suppose J is a subspace of A such that $J = (1-q)A^{**} \cap A$ for a closed projection $q \in A^{\perp\perp}$. The subspace J is a right ideal of A and $J = \{a \in A : qa = 0\}$. By Theorem 4.1, $J^{\perp\perp}$ contains $1-q$, which is a left identity for $J^{\perp\perp}$. Thus J possesses a left contractive approximate identity (see e.g. 2.5.8 in [8]). $\square$

5. PEAK PROJECTIONS

From the last section we see can see the role that closed projections in $A^{\perp\perp}$ play in determining the right ideal structure of an operator algebra A. In this section we study the closed projections in $A^{\perp\perp}$ from the view point of 'peak phenomena' in A. Indeed, this idea was already subtly playing a role in the proof of Theorem 4.1. The following theorem will be the basis for our definition of a noncommutative peak set, or *peak projection*. However, first note that if a is a contraction and q is a projection such that $qa = q$, then a and q necessarily commute.

Theorem 5.1. *Let a be a contraction in B and let q be a closed projection in B^{**} such that $aq = q$. Then the following are equivalent:*

(1) $\varphi(a^*a) < 1$ *for all* $\varphi \in S(B)$ *such that* $\varphi(q) = 0$,
(2) $\varphi(a^*a(1-q)) < 1$ *for all* $\varphi \in S(B)$,
(3) $\varphi(a^*a(1-q)) < \varphi(1-q)$ *for all* $\varphi \in S(B)$ *such that* $\varphi(1-q) \neq 0$,
(4) $\varphi(a^*a) < 1$ *for every pure state* φ *of* B *such that* $\varphi(q) = 0$,
(5) $\|pa\| < 1$ *for any closed projection* $p \leq 1-q$,
(6) $\|ap\| < 1$ *for any closed projection* $p \leq 1-q$, *and*
(7) $\|ap\| < 1$ *for any minimal projection* $p \leq 1-q$.

Proof. (3) $\Rightarrow$ (2) Assume (3) holds. If $\varphi \in S(B)$ is such that φ doesn't vanish on $1-q$, we have $\varphi(a^*a(1-q)) < \varphi(1-q) = 1 - \varphi(q) \leq 1$. In the case that φ vanishes on $1-q$, (2) follows by the Cauchy-Schwarz inequality for positive linear functionals.

(2) $\Rightarrow$ (3) Suppose (2) holds. Let φ be a state which does not vanish on $1-q$ and define

$$\psi(\cdot) = \frac{\varphi(\cdot(1-q))}{\varphi(1-q)}.$$

Because ψ is contractive and unital, it is a state on B. Applying ψ to both sides of $a^*a(1-q) < 1$ we get

$$\frac{\varphi(a^*a(1-q)(1-q))}{\varphi(1-q)} < 1,$$

which implies that $\varphi(a^*a(1-q)) < \varphi(1-q)$. So (3) holds.

(1) $\Rightarrow$ (4) This is immediate.

(2) $\Rightarrow$ (1) Assume (2) holds and let $\varphi \in S(B)$ be a state such that $\varphi(q) = 0$. Then $\varphi(a^*a) = \varphi(a^*a) - \varphi(q) = \varphi(a^*a(1-q)) < 1$. So (1) holds.

(4) $\Rightarrow$ (2) Assume (4) and let $\varphi \in S(B)$ and suppose that $\varphi(a^*a(1-q)) = 1$. It follows that $\varphi(a^*a) = 1$ and $\varphi(q) = 0$. Consequently, $\varphi(1 - a^*a) = 0$. Now let J be the left ideal $B^{**}(1-q) \cap B$ of B, so that $J \cap J^* = (1-q)B^{**}(1-q) \cap B$ is a hereditary subalgebra of B. Note that $1 - a^*a = (1-a^*a)(1-q)$ is an element of $J \cap J^*$. Let $(e_t) \subset J \cap J^*$ be an increasing contractive approximate identity for $J \cap J^*$. Then (e_t) is an increasing left contractive approximate identity for J, However, since $1-q$ is the support projection for J, then necessarily $e_t \to 1-q$ weak*, and so $\varphi(e_t) \to \varphi(1-q) = 1$. However,

$$\|\varphi|_{J\cap J^*}\| = \lim_t \varphi|_{J\cap J^*}(e_t)$$

and so $\|\varphi|_{J\cap J^*}\| = 1$, making $\varphi|_{J\cap J^*}$ a state of $J\cap J^*$ such that $\varphi|_{J\cap J^*}(1-a^*a) = 0$. By Lemma 2.5 there exists a pure state ψ of $J \cap J^*$ which annihilates $1 - a^*a$. We can then extend ψ to a pure state $\tilde{\psi}$ of B. We then have

$$0 = \tilde{\psi}(1 - a^*a) = \tilde{\psi}(1) - \tilde{\psi}(a^*a) = 1 - \tilde{\psi}(a^*a).$$

Thus $\tilde{\psi}(a^*a) = 1$. We also have

$$\begin{aligned} 1 &= \tilde{\psi}(1) = \tilde{\psi}(q) + \tilde{\psi}(1 - q) \\ &= \tilde{\psi}(q) + \lim_t \tilde{\psi}(e_t) \\ &= \tilde{\psi}(q) + \lim_t \psi(e_t) \\ &= \tilde{\psi}(q) + \|\psi\| = \tilde{\psi}(q) + 1, \end{aligned}$$

which forces $\tilde{\psi}(q) = 0$. Thus we have found a pure state on B which annihilates q, but takes the value 1 at a^*a. This contradicts our assumption of (4). Thus, the supposition that $\varphi(a^*a(1 - q)) = 1$ must be rejected, and therefore (2) holds.

(6) $\Rightarrow$ (7) This is immediate from the fact that any minimal projection is automatically closed.

(7) $\Rightarrow$ (4) Assume (7) and let φ be a pure state of B which annihilates q. Let φ be such a pure state. Let L be the left-kernel of φ. Then $L = B^{**}(1-p)\cap B$ for some minimal closed projection $p \in B^{**}$, and the weak*-closure of L in B^{**} will be $B^{**}(1-p)$ (3.13.6 in [21]). Now viewing φ as a normal state on B^{**}, let L' be the left-kernel of φ with respect to B^{**}. This will be a weak*-closed left ideal and $L' = B^{**}r$ for some projection $r \in B^{**}$ with $q \in L'$. The containment $B^{**}(1-p)\cap B \subset B^{**}r$ is obvious. Passing to the weak*-closure we get $B^{**}(1-p) \subset B^{**}r$. Hence, $1-p \leq r$, or rather $1 - r \leq p$, which by minimality of p forces $p = 1 - r$. We conclude that $\overline{L}^{w^*} = L'$ and therefore $q \in B^{**}(1-p)$. Hence $q \leq 1-p$, or equivalently, $p \leq 1-q$. Thus by condition (7), $\|ap\| < 1$. Now decompose a^*a as

$$a^*a = pa^*ap + pa^*a(1 - p) + (1 - p)a^*ap + (1 - p)a^*a(1 - p)$$

and apply φ to get

$$\varphi(a^*a) = \varphi(pa^*ap) + \varphi(pa^*a(1 - p)) + \varphi((1 - p)a^*ap) + \varphi((1 - p)a^*a(1 - p)).$$

The last 3 terms vanish by the Cauchy-Schwarz inequality and the fact that $\varphi(1 - p) = 0$. Thus we have

$$\varphi(a^*a) = \varphi(pa^*ap) \leq \|pa^*ap\| = \|ap\|^2 < 1,$$

establishing (4).

(2) $\Rightarrow$ (5) Assume (2) and let $p \leq 1-q$ be closed. Since p is closed, a^*pa is upper semi-continuous on $S(B)$. Therefore, a^*pa attains its maximum on $S(B)$ at some state φ, and hence $\|pa\|^2 = \varphi(a^*pa)$. Since $p \leq 1 - q$ we have $a^*pa \leq a^*(1 - q)a$, and so

$$\varphi(a^*pa) \leq \varphi(a^*(1 - q)a) = \varphi(a^*a(1 - q)) < 1,$$

by (2) and the fact that a and q commute. Thus, the norm of pa must be strictly less than 1.

(6) $\Rightarrow$ (7) Apply the implication (7) $\Rightarrow$ (6), which has already been established, to a^*. $\square$

Remarks First, the condition $\|ap\| < 1$ is equivalent to $\|pa^*\| < 1$, which means that a^*a in conditions (1)-(4) can be replaced with aa^*. Second, similarly to the equivalence of (6) and (7), condition (5) is equivalent to the statement that $\|pa\| < 1$ for any minimal projection $p \leq 1 - q$.

Definition 5.2. If A is a unital subalgebra of a unital C^*-algebra B, a projection $q \in B^{**}$ is called a *peak projection for A* if there exists a contraction $a \in A$ such that $qa = q$ and such that a and q satisfy the equivalent conditions of Theorem 5.1. We refer to a as the *peak associated with q*. If q is an intersection of peak projections, we refer to q as a *p-projection*.

If $B = C(K)$, the continuous complex-valued functions on a compact space K, then the peak projections for A are exactly the characteristic functions of peak sets. It is also easy to see that if E is a peak set and a is its associated peak, then a^n converges point-wise to χ_E. From Lemma 3.4, the same is true for peak projections.

As a consequence, we have the following proposition.

Proposition 5.3. *Any projection satisfying condition (1) in Theorem 5.1 for some contraction in a unital subspace A of a unital C^*-algebra B is a closed projection.*

Proof. Let A be a unital subspace of B and let q be a projection in B^{**} satisfying condition (1) of Theorem 5.1 for some contraction $a \in A$, then $(a^*a)^n$ is a decreasing net of self-adjoint elements in B with limit q. Hence q is closed. Any intersection of closed projections is again closed. Thus the result also holds for p-projections. □

For a compact Hausdorff space K, any closed set in K will be a p-set for $C(K)$ by Urysohn's lemma. The same thing holds in the noncommutative case as well.

Proposition 5.4. *For a C^*-algebra B, any closed projection in B^{**} is a p-projection for B.*

Proof. Let $q \in B^{**}$ be a closed projection. Then for any open projection $u \geq q$, there exists $a_u \in B$ with $0 \leq a_u \leq 1$ such that $a_u q = q$ and $a_u(1-u) = 0$. Now let q_u be the weak*-limit of a_u^n as $n \to \infty$. Since multiplication is separately weak*-continuous, it follows that $a_u q_u = q_u a_u = q_u$. From this property and again the separate weak*-continuity of multiplication, it follows that q_u is a projection. Since $a_u(1-u) = 0$, we also have $q_u \leq u$. We now claim that q_u is a peak projection for B with peak a_u. We only need to check that $\varphi(a_u^2) < 1$ for any pure state $\varphi \in B^*$ such that $\varphi(q_u) = 0$. However, since $a_u^n \to q_u$ weak*, it follows that $\varphi(a_u^n) \to 0$. Suppose that $\varphi(a_u^2) = 1$. Then representing B concretely on a Hilbert space so that φ is a vector state $\varphi(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle$, we see that $\langle \pi(a_u^2)\xi, \xi \rangle = 1$. So by the converse to the Cauchy-Schwarz inequality, we must have that $\pi(a_u^2)\xi = \xi$, which contradicts that $\varphi(a^n) \to 0$. So q_u is a peak projection. Note also that the equation $a_u^n q = q$ implies that $q \leq q_u$.

Now we take the intersection $\bigwedge q_u$ of all such q_u as u varies over all open projections dominating q. We now show that $q = \bigwedge q_u$. To see this, note that since $q \leq q_u$, we have that $q \leq \bigwedge q_u$. By Proposition 2.2,

$$q \leq \bigwedge q_u \leq \bigwedge u = q.$$

Thus, $q = \bigwedge q_u$. □

This next proposition describes a peak projection in terms of a support projection associated with its peak.

Proposition 5.5. *A projection $q \in B^{**}$ is a peak projection for a unital subspace A of B if and only if there exists a contraction $a \in A$ such that $1-q$ is the right support projection for $1-a$.*

Proof. We assume that B and B^{**} are acting on the universal Hilbert space H_u for B. First suppose that q is a peak projection for A with peak $a \in A$. Let $\xi \in H_u$ and let r denote the right support projection of $1-a$. Since $a^n \to q$ weak*, $aq = q$ implies that $\xi \in \text{Ran } q$ if and only if $a\xi = \xi$. This in turn is equivalent to saying $\xi \in \text{Ran } q$ if and only if $\xi \in \text{Ker } (1-a)$. Since r is the projection onto $[\text{Ker } (1-a)]^{\perp}$, this last statement is equivalent to $1-q = r$.

Now suppose that $a \in A$ is a contraction and q is a closed projection such that the range projection r, of $1-a$, is equal to $1-q$. Then $1-a = (1-a)r = (1-a)(1-q)$ implies $aq = q$. Now define $b = (a+1)/2$, so that $bq = q$. Let φ be a state on B such that $\varphi(q) = 0$. We may assume that $\varphi(\cdot) = \langle \cdot \eta, \eta \rangle$ for a unit vector $\eta \in H_u$. Then $q\eta = 0$, and so $r\eta = \eta$. Now suppose that $\varphi(b^*b) = 1$. Then

$$1 = \varphi(b^*b) = \frac{1}{4}(\varphi(a^*a) + 2\text{Re } \varphi(a) + 1).$$

Hence, $1 = \varphi(a^*a) = \langle a\eta, \eta \rangle$, and so by the converse to the Cauchy-Schwarz inequality, $a\eta = \eta$. Thus $\xi \in \text{Ker } (1-b)$, and so $r\eta = 0$, which is a contradiction. Thus $\varphi(b^*b) < 1$ and so q is a peak projection with peak b. $\square$

In the commutative case, it is easy to see that if E and F are peak sets for a uniform algebra A with peaks f and g, respectively, in A. then $E \cap F$ will be a peak set with peak $\frac{1}{2}(f+g)$. For general C^*-algebras, we have the following generalization.

Proposition 5.6. *Let q_1 and q_2 be two peak projections with peaks a_1 and a_2, respectively. Then $q_1 \wedge q_2$ is also a peak projection with peak $\frac{1}{2}(a_1 + a_2)$.*

Proof. That $q_1 \wedge q_2$ and $\frac{1}{2}(a_1 + a_2)$ satisfy the first condition in the definition of peak projection is immediate since $q_1 \wedge q_2$ is dominated by both q_1 and q_2. To show the second condition, let φ be a pure state of B which annihilates $q_1 \wedge q_2$. Let $b = \frac{1}{2}(a_1 + a_2)$. We wish to show $\varphi(b^*b) < 1$. This is equivalent to showing

$$\varphi(a_1^*a_1) + \varphi(a_1^*a_2) + \varphi(a_2^*a_1) + \varphi(a_2^*a_2) < 4,$$

for which it suffices to show that either $\varphi(a_1^*a_1) < 1$ or $\varphi(a_2^*a_2) < 1$. So suppose $\varphi(a_1^*a_1) = \varphi(a_2^*a_2) = 1$. Let $\pi : B^{**} \to B(H)$ be the weak*-continuous cyclic representation associated with φ and let $\xi \in H$ be the corresponding cyclic vector. Then

$$\langle \pi(a_1^*a_1)\xi, \xi \rangle = \varphi(a_1^*a_1) = 1.$$

Thus, by the converse to the Cauchy-Schwarz inequality, we must have $\pi(a_1^*a_1)\xi = \xi$. However, since $(a_1^*a_1)^n \to q_1$ weak*, and so $\pi(a_1^*a_1)^n \to \pi(q_1)$ in the weak operator topology, we must have $\varphi(q_1) = 1$. The same argument shows that $\varphi(q_2) = 1$ and so ξ must be in the range of both $\pi(q_1)$ and $\pi(q_2)$. Hence, $\varphi(q_1 \wedge q_2) = \langle \pi(q_1) \wedge \pi(q_2)\xi, \xi \rangle = 1$, which is a contradiction. $\square$

The next result shows that p-projections must be in the weak* closure of A in B^{**} when A is a subalgebra.

Proposition 5.7. *Let A be a unital subalgebra of a unital C^*-algebra B, and let q be a p-projection for A. Then $\varphi(q) = 0$ for all φ in $A^\perp$. Consequently, φ is in $(qA)_\perp$ and q is in $A^{\perp\perp}$.*

Proof. First assume q is a peak projection and let $a \in A$ be the peak associated with q. For any φ in $A^\perp$ and any integer $n > 0$, $\varphi(a^n) = 0$. However, a^n converges weak* to q. Thus $\varphi(q) = 0$.

Now suppose $q = \bigwedge_i q_i$ and let $\varphi \in A^\perp$. Let $\epsilon > 0$. By a result in [1] there exists an open projection $p \in B^{**}$ such that $p \geq q$ and $|\varphi|(p - q) < \epsilon$, where $|\varphi|$ is obtained from the polar decomposition of φ (3.6.7 in [21]). By hypothesis $\bigwedge_i q_i \leq p$. Hence $1 - p \leq \bigvee_i (1 - q_i)$, and so by the compactness property of closed projections there, exist finitely many projections $q_1, q_2, \ldots, q_n$ in the family $\{q_i\}$ such that $1 - p \leq \bigvee_{i=1}^n (1 - q_i)$. Thus $q \leq \bigwedge_{i=1}^n q_i \leq p$. Now let $Q = \bigwedge_{i=1}^n q_i$, which is again a peak projection. By the last paragraph it follows that $\varphi(Q) = 0$, and so $|\varphi(Q - q)| = |\varphi(q)|$ for all $\varphi \in A^\perp$. The functional $|\varphi|$ has the property that $|\varphi(x)|^2 \leq \|\varphi\||\varphi|(x^*x)$ for all $x \in B^{**}$. Thus we have

$$\begin{aligned} |\varphi(Q-q)|^2 &\leq \|\varphi\||\varphi|((Q-q)^*(Q-q)) \\ &= \|\varphi\||\varphi|(Q-q)) \\ &\leq \|\varphi\||\varphi|(p-q)) \\ &< \|\varphi\|\epsilon. \end{aligned}$$

Since ϵ was arbitrary, this shows that $\phi(q) = 0$.

If $x \in A$, the map $\varphi(\cdot x)$ is also in $A^\perp$. Thus $\varphi(qx) = 0$ for every x in A. This shows that φ is in $(qA)_\perp$. By Lemma 2.4, this implies that q is in $A^{\perp\perp}$. $\square$

It is natural to ask if the notion of a peak or p-projection is dependent on the particular C^*-algebra in which we view A as residing. That is, if we have embeddings of A into two different C^*-algebras, can the peak projections arising from the each embedding be identified in some way? By an embedding of A into a C^*-algebra, we mean a completely isometric homomorphism of A. The following proposition shows that the notion of a peak or p-projection is indeed independent of the particular embedding.

Proposition 5.8. *Let A be a unital subalgebra of a C^*-algebra B. Let $\pi : A \to B_1$ be a unital completely isometric homomorphism of A into another C^*-algebra B_1. If q is a p-projection for A in B^{**}, then $\pi^{**}(q)$ is a p-projection for $\pi(A)$ inside B_1^{**}.*

Proof. We assume that B_1 is acting on its universal Hilbert space H_u, that is $B_1 \subset B(H_u)$. Assume that q is a peak projection with peak $a \in A$. Clearly, $\pi(a)\pi^{**}(q) = \pi^{**}(q)$. Now suppose that $\varphi \in S(B_1)$ such that $\varphi(\pi^{**}(q)) = 0$ and extend φ to a vector state $\tilde{\varphi}$ on $B(H_u)$. Viewing π as a unital completely positive map into $B(H_u)$, by Arveson's extension theorem we may extend π to a completely positive map $\rho : B \to B(H_u)$. This in turn can be extended to a weak*-continuous map $\tilde{\rho}$ on B_1^{**}. Since π^{**} is the unique weak*-continuous extension of π to A^{**}, we must have $\tilde{\rho}|_{A^{**}} = \pi^{**}$. We next observe that $\tilde{\varphi} \circ \rho$ is a state on B which extends uniquely to a weak*-continuous state on B^{**}, which by uniqueness must be $\tilde{\varphi} \circ \tilde{\rho}$. Hence, $(\tilde{\varphi} \circ \rho)(q) = \tilde{\varphi}(\tilde{\rho}(q)) = \varphi(\pi^{**}(q)) = 0$. Thus, since q is a peak projection, we must have that $\tilde{\varphi}(\rho(a^*a)) < 1$. By the Kadison-Schwarz inequality for completely

positive maps, we have

$$\varphi(\pi(a)^*\pi(a)) = \tilde{\varphi}(\rho(a)^*\rho(a)) \leq \tilde{\varphi}(\rho(a^*a)) < 1.$$

Thus $\pi^{**}(q)$ is a peak projection for $\pi(A)$. Now if q is just a p-projection with $q = \wedge q_i$ for peak projections q_i, then q is the weak* limit of the net of meets for finitely many q_i. Thus, by the discussion about meets in Section 2, it follows that $\pi^{**}(q) = \wedge \pi^{**}(q_i)$. Thus $\pi^{**}(q_i)$ is a p-projection. $\square$

Minimal projections which are also p-projections correspond to *p-points* (an intersection of singleton p-sets) in the commutative case. The closure of the set of p-points for a uniform algebra is the Shilov boundary (see e.g. [13] and [23]). Let A and B be as before, but assume A generates B as a C^*-algebra. Then there exists a largest closed two-sided ideal J of B such that the canonical quotient map $B \to B/J$ restricts to a complete isometry on A ([3]). The ideal J is the so-called 'Shilov ideal' for A. Let p be the closed projection in B^{**} corresponding to the Shilov boundary ideal, then p dominates all minimal projections which are also p-projections for A. Moreover, p dominates all minimal projections in $A^{\perp\perp}$ (see [7]).

Unfortunately, however, the join of the orthogonal complements of all such minimal projections does not in general equal the support of the Shilov ideal.

One of the interesting aspects of p-projections is their relationship to approximate identities. For instance, we have the following proposition ([16],[7]).

Proposition 5.9. *If A is a unital subalgebra of a C^*-algebra B and $p \in B^{**}$ is the support projection for a right ideal of A with a left approximate identity of the form $(1 - x_t)$ for $\|x_t\| \leq 1$, then $1 - p$ is a p-projection for A.*

Proof. Let J be a right ideal of A with left approximate identity (e_t) with $e_t = 1-x_t$ with x_t in the unit ball of A. Let p be its support projection and define $q = 1 - p$. Then q is necessarily closed, $e_t \to 1 - q$ weak*, $J = \{a \in A : (1 - q)a = a\}$, and $(1 - e_t)q = q$. For each t let J_t be the intersection of all right ideals of B containing e_t. Then there exists a unique closed projection q_t in B^{**} such that $J_t = (1 - q_t)B^{**} \cap B$. By the proof of Lemma 3.5 q_t is a peak projection with peak $\frac{1}{2}[1 + (1 - e_t)] = 1 - \frac{1}{2}e_t$ and such that $q_t \geq q$. Now set $r = \wedge q_t$. We have that $r \geq q$, but suppose $r - q \neq 0$. Then $(r - q)e_t \to (r - q)(1 - q) = r - q$, and $r - q \leq q_t$. Thus $(1 - q_t)B^{**} \subset (1 - (r - q))B^{**}$, so that $(e_t) \subset (1 - (r - q))B^{**}$. Hence $(1-(r-q))e_t = e_t$ for each t, and so $(r-q)e_t = 0$. However, $(r-q)e_t \to r-q$. Thus $r - q = 0$ and so $r = q$, making q an intersection of peak projections. $\square$

Remark It can also be shown that if an ideal J of a unital operator algebra A has a left contractive approximate identity, then it has a left approximate identity of the form $(1 - x_t)$, where $x_t \in A$ and $\lim_t \|x_t\| = 1$ ([7]). Moreover, if we can choose the x_t in $\mathrm{Ball}(A)$, for every such ideal, then the p-projections are exactly the orthogonal complements of the support projections for right ideals with left contractive approximate identity.

It is natural to make the following definition.

Definition 5.10. Let A be a unital subalgebra of a unital C^*-algebra B. A projection $q \in B^{**}$ is said to be an *approximate p-projection for A* if q is closed and $q \in A^{\perp\perp}$.

The following shows that approximate p-projections possess peaking properties.

Theorem 5.11. *Let A be a unital subalgebra of a unital C^*-algebra B and let $q \in B^{**}$ be a closed projection. Then the following are equivalent:*

1. *q is an approximate p-projection,*
2. *for every $\epsilon > 0$ and for every open projection $u \geq q$, there exists $a \in (1+\epsilon)Ball(A)$ such that $qa = q$ and $\|a(1-u)\| \leq \epsilon$, and*
3. *for every $\epsilon > 0$ and strictly positive $p \in B$ with $p \geq q$, there exists $a \in A$ such that $qa = q$ and $a^*a \leq p + \epsilon$.*

Proof. (1) $\Rightarrow$ (3) This is essentially Lemma 3.2. Let $\epsilon > 0$ and let $p \in B$ be a strictly positive element of B such that $p \geq q$. Since q is in $A^{\perp\perp}$, by Lemma 2.4, q satisfies the hypothesis of Lemma 3.2. Thus there exists $a \in A$ such that $qa = q$ and $a^*a \leq p + \epsilon$.

(3) $\Rightarrow$ (2) Let $\epsilon > 0$ and suppose $u \geq q$ is open. Let $\delta = \epsilon^2/2$. As we have seen before, by the noncommutative Urysohn's lemma, there exists a strictly positive contraction $p \in B$ such that $pq = q$ and $p(1-u) = \delta(1-u)$. By (3) there exists $a \in A$ such that $qa = q$ and $a^*a \leq p + \delta$. Hence, $(1-u)a^*a(1-u) \leq 2\delta(1-u)$, and so $\|a(1-u)\| \leq \epsilon$.

(2) $\Rightarrow$ (1) Let $u \geq q$ be an open projection. By (2), for each natural number n there exists $a_n \in (1+1/n)\mathrm{Ball}(A)$ such that $qa_n = q$ and $\|a(1-u)\| \leq \frac{1}{n}$. The net (a_n) has a weak* limit point $a \in \mathrm{Ball}(A^{\perp\perp})$. Since $\|(1-u)a_n\| \leq \frac{1}{n}$ for each n, we must also have $\|a(1-u)\| = 0$, and hence $au = a$. Let $b = \frac{1}{2}(a+1)$, which is in $A^{\perp\perp}$. By Lemma 3.6, there exists a projection $q_u \in A^{\perp\perp}$ such that $q \leq q_u$ and $b^k \to q_u$. As in the proof of Theorem 4.1 we also that $q \leq q_u \leq u$ for each u. However, by Proposition 2.2, this implies that $q = \wedge q_u$ as u varies over all open projections dominating q. However, each q_u is in $A^{\perp\perp}$. Thus, so is q, by the discussion in Section 2. □

For a uniform algebra $A \subset C(K)$, Glickberg's peak set theorem says that a closed set E is a p-set for A if and only if $\mu \in A^\perp$ implies $\mu_E \in A^\perp$. With this and Lemma 2.4 in mind, it is then natural to ask whether or not the p-projections are precisely the approximate p-projections, in the noncommutative setting. Certainly any p-projection is an approximate p-projection. The reverse implication holds for unform algebras by the classical Glicksberg theorem, and it holds when $A = B$ is a C^*-algebra by Proposition 5.4. It is also true for operator algebras which are also reflexive Banach spaces, as the following simple observation shows. In particular, it is true for finite dimensional algebras.

Proposition 5.12. *Let A be a unital subalgebra of a unital C^*-algebra B such that A is also a reflexive Banach space. Let $q \in B^{**}$ be a closed projection. The following are equivalent:*

1. *q is a p-projection for A,*
2. *q is an approximate p-projection for A, and*
3. *$q \in A$.*

Proof. The implication (1) $\Rightarrow$ (2) follows from Proposition 3.4 and the fact that A is an algebra. If (2) holds, by reflexivity, q is in A, establishing (3). If q is in A, then it is trivially a p-projection. So (3) $\Rightarrow$ (1) holds. □

Approximate p-projections enjoy some of the properties of p-projections. For example, by some observations in Section 2, if A is a unital subalgebra of a unital C^*-algebra B, then the meet of a collection of approximate p-projections for A is also an approximate p-projection. It can also be shown that if the join of a collection of approximate p-projections happens to be closed, then it is also an approximate p-projection (see [16]). By Corollary 5.5 of [7], if $q \in B^{**}$ is a closed projection and X is a unital subspace of B such that for every strictly positive contraction $p \in B$ with $q \leq p$ there exists $a \in X$ such that $aq = q$ and $a^*a \leq p$, then q is a p-projection for X.

As described above, for many algebras the class of p-projections is the same as the class of approximate p-projections. The most tantalizing remaining question here is whether or not these two notions are the same for a general unital operator algebra. Nonetheless, the correspondence between right ideals with left approximate identity and approximate p-projections will be key in importing some results from C^*-algebra theory to general operator algebras. Indeed, we have begun this in [7].

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